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Scheme Dependence and Equivalence of Sensitivity for Nonlinear Problems

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In static nonlinear structural analysis, there are several schemes widely employed to calculate the responses of structures. Examples are the Newton–Raphson method, displacement control method, arc length method, and work control method. It is known that the responses obtained are independent of the schemes chosen for the analysis problem. It is shown in this paper that if sensitivities of response variables are computed, different schemes do give rise, in general, to different results. However, these sensitivities are all equivalent. A systematic approach is derived in order to show this equivalence. This approach is derived in incremental form using the analytical differentiation approach. Numerical examples for validating the present approach include a nonlinear elastic spring model and geometric nonlinear analysis of trusses. The finite element method is used in the truss examples. Interestingly, it is also found that the finite difference method performs poorly in obtaining the sensitivities corresponding to the arc length and work control methods, especially for structures with limit point response.

Introduction

SENSITIVITY analysis of nonlinear problems has attracted great attention in the past few years. These problems, typically, involve geometrical and/or material nonlinearities.[1–8] Also, to formulate the sensitivity equations, one of three approaches is often used[9,10]: the direct differentiation approach (DDA), the adjoint structure approach (ASA), and the finite difference approach (FDA). Typically, these three methods are used in conjunction with numerical techniques such as the finite element method (FEM)[1–4,8] or the boundary element method (BEM).[5–7] In the present study, the DDA and the FEM are adopted. Here, the focus is on sensitivity analysis for structures with geometric nonlinearities. Material nonlinearities with path-independent characteristics are also considered.

Some of the recent research on geometric nonlinear sensitivity analysis has emphasized the calculation of sensitivity of bifurcation and limit loads. The reader is referred to Refs. 2 and 3 for a discussion of this subject. However, the focus of the present study is on the scheme dependence of sensitivity and its equivalent transformation. In static nonlinear structural analysis, a robust numerical scheme is needed to determine the response of structures. The commonly used numerical schemes include the Newton–Raphson method (where load steps are specified or controlled), the displacement control method, the arc length method, and the work control method. Among them, the arc length and work control methods are suitable for problems with snap-through and/or snap-back responses; the displacement control method can be used to trace the response with snap-through but not snap-back behavior, whereas the Newton–Raphson method fails to trace even snap-through response (e.g., Refs. 11 and 12). It is known that the structural responses obtained using different schemes give different points on the same solution curve. Thus, the simulation results are scheme independent.

It is interesting, however, that the sensitivity obtained from the different schemes are, in general, different. This apparent discrepancy in the calculated sensitivities is not a consequence of the choice of a particular numerical scheme. Rather, the adopted constraint equation, although not affecting the calculated structural response, does strongly affect the sensitivity results.

This situation is analogous to the effect of different, but equivalent, prescribed boundary conditions on sensitivity results for continua. Consider, for example, the one dimensional deformation of an elastic bar of length L, cross-sectional area A, and Young's modulus E. Of course, an applied axial load P or an applied end displacement $\Delta = PL/AE$ gives the same stress σ in the bar. However, with L as the design variable, the stress sensitivities, $\overset{*}{\sigma} = \mathrm{d}\sigma/\mathrm{d}L$, are quite different, i.e., for prescribed P, $\overset{*}{\sigma} = 0$, and for prescribed $\Delta, \overset{*}{\sigma} = -E\Delta/L^2$.

A natural question to ask here is: are these different sensitivities, in some sense, equivalent? Can one directly transform the sensitivities from one situation to another? This would be useful, for example, if one needs the sensitivity history for a prescribed load problem but could perform, due to certain physical constraints, only displacement controlled experiments.

Returning now to the structural mechanics problem, consider the snap-through response of a two member truss (see Fig. 1) of initial height h. Let h be the design variable and F be the axial force in a certain member of the truss. Suppose two displacement controlled snap through experiments are performed on the truss, with slightly different values of h, and the sensitivity history $\mathrm{d}F/\mathrm{d}h$ is obtained by the finite difference method. These sensitivities can also be calculated by the displacement control algorithm. Now, suppose that one is required to know the sensitivity history of F, with respect to h, when, instead, the load history is prescribed on the truss. Now one has a problem. The load-controlled experiments cannot provide any data in the unstable region following snap through. Also, the Newton–Raphson (load control) method for calculation of these sensitivities fails following snap through. Of course, one could calculate the sensitivity history for this physical situation by any of the other methods, i.e., displacement control, arc length, or work control. In general, the results are all different even though the same physical situation is being modeled. Which, if any, of these numerical results is correct?

On purely physical grounds, one expects that the sensitivities, calculated from different schemes, must be equivalent in the sense that (see Fig. 2) given the response to the nominal design and any sensitivity history, one must be able to construct the same response of the perturbed design. The central contribution of this paper is to show that these sensitivities obtained from different schemes are, indeed, equivalent. A new approach is proposed here to demonstrate this equivalence.

Spring Model

In this section, a simple spring model with nonlinear elastic material behavior is used to illustrate the themes of this study. Assume that the constitutive law of an elastic spring is described by

$$p = k_0 x + k_1 x^3 \tag{1}$$

where p is the spring force and x the elongation.

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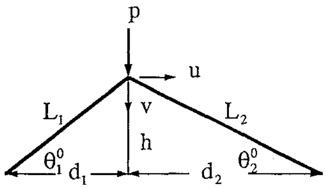

Fig. 1 Shallow truss.

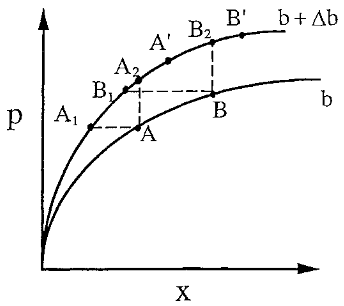

Fig. 2 Design mappings from states A, B to A', B' depend on the constraint associated with a chosen scheme; for example, for NR, mapped points are A_1B_1 and for DC, they are A_2B_2.

The response curve of p vs x for the model can be obtained by the various numerical schemes mentioned earlier. The responses are always the same.

An incremental form of Eq. (1), form step n to step $n+1$, can be written as

$$\left(k_0 + 3k_1x_n^2\right)\Delta x + k_1\left(3x_n\Delta x^2 + \Delta x^3\right) - \Delta p = 0 \tag{2a}$$

or

$$(k_t + k_h)\Delta x - \Delta p = 0 \tag{2b}$$

with Δp and Δx denoting the increments of p and x, respectively, and

$$k_t(x_n) = k_0 + 3k_1x_n^2,$$

$$k_h(x_n, \Delta x) = k_1\left(3x_n\Delta x + \Delta x^2\right)$$

denoting the tangent and higher order stiffness terms, respectively.

There are two unknowns Δx and Δp in Eq. (2a) or (2b). Thus, in addition to Eq. (2a) or (2b), one more equation is needed to solve for Δx and Δp uniquely. For the Newton–Raphson (NR) method, this equation is Δp = a specified load increment. For the displacement control (DC) method, it is Δx = a specified displacement increment. For the arc length (ARC) method (where the spherical arc length method is used throughout this study,[12]) the constraint is

$$\sqrt{\Delta x^2 + \Delta p^2} = \Delta S \tag{3}$$

with ΔS being a specified arc length, and for the work control method

$$\Delta x \Delta p = \Delta W \tag{4}$$

with ΔW being a specified work increment (more precisely, twice the work increment).

In the following, sensitivity with respect to k_0 and k_1 is studied for this model. The equations governing sensitivities of Δx and Δp can be obtained by the direct differentiation approach (e.g., Refs. 9 and 10). First, the sensitivity version of Eq. (2a) is

$$k_c\,\Delta\overset{*}{x} - \Delta\overset{*}{p} = -\overset{*}{k}_0\Delta x - \overset{*}{k}_1\left(3x_n^2\Delta x + 3x_n\Delta x^2 + \Delta x^3\right)$$
$$- 3k_1\overset{*}{x}_n\Delta x(2x_n + \Delta x) \tag{5}$$

where k_c is defined as

$$k_c = \left(k_0 + 3k_1x_n^2\right) + k_1\left(6x_n\Delta x + 3\Delta x^2\right) \tag{6}$$

and an asterisk denotes the design derivative of a variable with respect a design variable b (like a material derivative). For example, $\Delta\overset{*}{x} = d(\Delta x)/db$. Here, two cases are considered: case 1, $b = k_0$, then $\overset{*}{k}_0 = 1$ and $\overset{*}{k}_1 = 0$; and case 2, $b = k_1$, then $\overset{*}{k}_0 = 0$ and $\overset{*}{k}_1 = 1$. Next, the sensitivity versions of the constraint equations are as follows.

NR:

$$\Delta\overset{*}{p} = 0 \tag{7a}$$

DC:

$$\Delta\overset{*}{x} = 0 \tag{7b}$$

ARC:

$$\Delta x\Delta\overset{*}{x} + \Delta p\Delta\overset{*}{p} = 0 \tag{7c}$$

WC:

$$\Delta\overset{*}{x}\Delta p + \Delta x\Delta\overset{*}{p} = 0 \tag{7d}$$

The solution procedure from state n with values $(x_n, p_n, \overset{*}{x}_n, \overset{*}{p}_n)$ to state $n+1$ is as follows. First, solve iteratively the nonlinear algebraic incremental equilibrium equation Eq. (2a) or (2b) together with the corresponding constraint equation associated with a chosen scheme, to obtain Δx and Δp. The iterative process continues until the residual force is smaller than a prescribed tolerance. Next, solve the sensitivity equation Eq. (5) together with Eq. (7a), (7b), (7c), or (7d) to obtain $\Delta\overset{*}{x}$ and $\Delta\overset{*}{p}$.

Before addressing the numerical results, the analytical solutions are given first for the displacement control and Newton–Raphson methods. These can be obtained by the use of the sensitivity versions of Eq. (1) and the corresponding constraint equation ($\overset{*}{x} = 0$ for the DC and $\overset{*}{p} = 0$ for the NR, respectively).

Case 1, DC:

$$\overset{*}{x} = 0, \qquad \overset{*}{p} = x \tag{8a}$$

Case 1, NR:

$$\overset{*}{p} = 0, \qquad \overset{*}{x} = -\frac{x}{k_0 + 3k_1x^2} \tag{8b}$$

Case 2, DC:

$$\overset{*}{x} = 0, \qquad \overset{*}{p} = x^3 \tag{9a}$$

Case 2, NR:

$$\overset{*}{p} = 0, \qquad \overset{*}{x} = -\frac{x^3}{k_0 + 3k_1x^2} \tag{9b}$$

In the numerical results [based on Eqs. (5) and (7a), (7b), (7c), or (7d) reported subsequently, $k_0 = 1$ and $k_1 = 0.1$ are adopted. Also, $\Delta p = 5$, $\Delta S = 5$, and $\Delta W = 2$ are used for the Newton–Raphson, arc length, and work control methods, respectively, such that all of the methods require approximately the same number of increments. The numerical solutions for the Newton–Raphson method and the displacement control method are in perfect agreement with the analytical solutions previously given and will be presented later using an equivalent approach proposed in this study. The numerical results obtained by using the arc length method is shown in Fig. 3 with different choices of ΔS. Note that since the iterative process for the arc length method is not carried out under a fixed loading or displacement, both $\overset{*}{x}$ and $\overset{*}{p}$ are nonzero (here, only $\overset{*}{p}$ is shown). Also, it is seen from Fig. 3 that the sensitivities are dependent on ΔS. The dependence of the sensitivity on ΔS arises from the use of a constraint equation which is a function of the finite increments Δx and Δp. More precisely, from Eq. (7c), one obtains

$$\frac{\Delta\overset{*}{p}}{\Delta\overset{*}{x}} = -\frac{\Delta x}{\Delta p} \tag{10}$$

It is clear that the ratio of $\Delta\overset{*}{p}$ and $\Delta\overset{*}{x}$ depends on the ratio of Δx and Δp, which, in turn, depends on the choice of ΔS [see Eq. (3)].

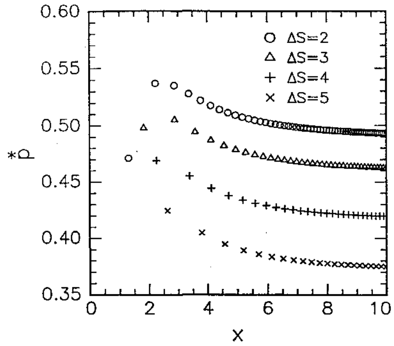

Fig. 3 Sensitivities depend on the arc length ΔS.

Similarly, the results from the work control method also depend on ΔW, and both $\Delta\overset{*}{p}$ and $\Delta\overset{*}{x}$ are nonzero; these results will be shown later using an equivalent approach.

A better understanding of the dependence of sensitivities, in the schemes employed, can be gained from the schematic Fig. 2, which shows a typical incremental step for both the nominal design and a perturbed design. In this figure, A and B denote states n and $n+1$, respectively, for the nominal design, and A' and B' (will be identified later as A_1 and B_1 for the NR and A_2 and B_2 for the DC) are the corresponding states associated with a perturbed design. For the displacement control method, the mapping from A or B to A_2 or B_2, respectively, has no changes in x; therefore $\overset{*}{x}$ is zero and $\overset{*}{p}$ is defined as the limit of $\Delta p/\Delta b$ at A or B. For the Newton–Raphson method, the corresponding mapped points are A_1 and B_1 and thus have no changes in p. Namely, $\overset{*}{p}$ is zero, and $\overset{*}{x}$ is calculated by $\Delta x/\Delta b$ at A or B. For the arc length or work control method, the mapped points A' and B', in general, have changes in both x and p; therefore, both $\overset{*}{x}$ and $\overset{*}{p}$ are nonzero. Suppose that the states A, B, and A' (solution of the previous step) have been determined; then the state B' can be determined by the corresponding constraint. For the arc length method, the constraint requires that the chord AB be equal to the chord $A'B'$. For the work control method, the constraint requires that the work increment from A to B be the same as that from A' to B'. Although the sensitivities are different for each scheme, based on physical grounds, use of any set of sensitivities should be able to construct the perturbed response in Fig. 2 if the response for the nominal design is given. Namely, they must be equivalent.

Now, an approach is developed here to show this equivalence. The basic philosophy is to transform all of the sensitivities to an equivalent solution corresponding to a certain scheme, say, the displacement control method. Then those transformed sensitivities will be called the equivalent displacement control sensitivities, and they should be the same as those directly calculated from the displacement control method.

First, the equilibrium equation Eq. (1) is rewritten as

$$f(x, p; k_0, k_1) = p - k_0 x - k_1 x^3 = 0 \tag{11}$$

Then, differentiation of Eq. (11) with respect to k_0 or k_1 yields

$$\overset{*}{f} = \frac{\partial f}{\partial p}\overset{*}{p} + \frac{\partial f}{\partial x}\overset{*}{x} + \frac{\partial f}{\partial b} = 0 \tag{12}$$

where b stands for k_0 or k_1. Evaluating the equations for two schemes $M1$ and $M2$ at the same state (x_n, p_n) and noting the fact that $\partial f/\partial b$ is at most a function of only x, p, and b but not of $\overset{*}{x}$, or $\overset{*}{p}$, one concludes

$$\left(\frac{\partial f}{\partial p}\overset{*}{p} + \frac{\partial f}{\partial x}\overset{*}{x}\right)M1 = \left(\frac{\partial f}{\partial p}\overset{*}{p} + \frac{\partial f}{\partial x}\overset{*}{x}\right)M2 \tag{13}$$

For example, if the sensitivities from all schemes are going to be transformed to an equivalent solution from the displacement control method (here called $M1$), then one uses the fact that for the $M1$ scheme $\overset{*}{x} = 0$. Since $\partial f/\partial p = 1$ from Eq. (11), one gets

$$(\overset{*}{p})_{M1} = \left(\frac{\partial f}{\partial p}\overset{*}{p} + \frac{\partial f}{\partial x}\overset{*}{x}\right)_{M2} \tag{14}$$

The results from the Newton–Raphson method (here called M_2) can be transformed analytically; since $\overset{*}{p} = 0$ for M_2 in Eq. (14), one obtains

$$(\overset{*}{p})_{M1} = -(k_0 + 3k_1 x^2)_{M2}(\overset{*}{x})_{M2} \tag{15}$$

Substituting the solutions of $\overset{*}{x}$ obtained in Eq. (8b) or (9b) for the Newton–Raphson method into Eq. (15), one recovers the same solutions for the displacement control method, as given in Eq. (8a) or (9a), respectively.

The equivalent displacement control results numerically obtained for all of the schemes are not shown here. However, they are in perfect agreement with the analytical solutions given in Eqs. (8a) and (9a), for cases 1 and 2, respectively. These results are obtained by substituting the solutions $(x_n, p_n, \overset{*}{x}_n, \overset{*}{p}_n)$, computed from each scheme into the right-hand side of Eq. (14), to obtain the equivalent displacement control $\overset{*}{p}_n$.

Note that the approach [Eq. (13)] can only be applied to obtain the equivalent Newton–Raphson or displacement control results. It cannot be used to relate the results from the arc length and work control methods, or to transform the results from the displacement control or Newton–Raphson method to those from the arc length or work control method. For this purpose, the constraint equations need to be considered. A general approach which can be used to relate sensitivities of any two schemes will be presented in the next section.

Before presenting the new approach, two points should be emphasized. First, Eqs. (5) and (7a–7d) are linear in terms of $\Delta\overset{*}{x}$ and $\Delta\overset{*}{p}$. Indeed, the equations governing sensitivities are always linear in sensitivities for nonlinear problems, even for path-dependent problems (e.g., Ref. 8). Therefore, no iterations are needed for solving the sensitivity equations.

Second, k_c defined in Eq. (6) is similar to the consistent tangent operator used in plasticity (path-dependent problems).[8] Namely k_c in Eq. (6) can be defined by

$$k_c(x_n, \Delta\tilde{x}) = \left.\frac{\partial \Delta p}{\partial \Delta x}\right|_{(x_n,\Delta\tilde{x})} = \left.\left(k_t + \frac{\partial(k_h \Delta x)}{\partial(\Delta x)}\right)\right|_{(x_n,\Delta\tilde{x})} \tag{16}$$

where $\Delta\tilde{x}$ denotes any intermediate value of Δx during iterations and the last identity is derived by using Eq. (2b). It is noted that for sensitivity analysis, k_c is only calculated once when the iterative process converges, i.e., at $\Delta\tilde{x} = \Delta x$.

The present paper is only concerned with path-independent problems, and the preceding paragraph is given for the sake of completeness. For such problems only, one can show that

$$k_t(\tilde{x}_{n+1}) = k_c(x_n, \Delta\tilde{x}) \tag{17}$$

(where $\tilde{x}_{n+1} = x_n + \Delta\tilde{x}$), so that it is sufficient to use the last tangent stiffness in the sensitivity calculations reported later in this paper.

Equivalent Transformation of Sensitivity

The equilibrium equations for static nonlinear structural analysis can be written as[10]

$$\boldsymbol{G}(\boldsymbol{u}; b) = \lambda(b)\boldsymbol{P}_r \tag{18}$$

where $\boldsymbol{G}$ is the internal force vector, $\boldsymbol{u}$ the displacement vector, b a design variable, $\boldsymbol{P}_r$ a reference load vector, and λ a scalar load factor. The dimensions of $\boldsymbol{G}$, $\boldsymbol{u}$, and $\boldsymbol{P}_r$ are N for a problem with N degrees of freedom. Here, proportional and conservative loading is assumed. Also, without loss of generality, only one design variable is discussed. Note that the reference load vector is a specified vector and should not be a function of b, i.e., $\overset{*}{\boldsymbol{P}}_r = 0$.

The incremental equilibrium equations can be written as

$$\Delta\boldsymbol{G}(\boldsymbol{u}_n; b) = \boldsymbol{G}(\boldsymbol{u}_n + \Delta\boldsymbol{u}; b) - \boldsymbol{G}(\boldsymbol{u}_n; b) = \Delta\lambda\boldsymbol{P}_r \tag{19}$$

where $\Delta\boldsymbol{u} = \boldsymbol{u}_{n+1} - \boldsymbol{u}_n$ and $\Delta\lambda = \lambda_{n+1} - \lambda_n$. Expressed in matrix form, Eq. (19) becomes

$$(\boldsymbol{K}_e + \boldsymbol{K}_g + \boldsymbol{K}_h)\Delta\boldsymbol{u} = \Delta\lambda\boldsymbol{P}_r \tag{20}$$

with K_e denoting the linear stiffness matrix, K_g the geometric stiffness matrix, and K_h the higher order stiffness matrix.[13,14] Usually, the sum of K_e and K_g is called the tangent stiffness matrix K_t, which is the Jacobian of G, i.e.,

$$K_e + K_g = K_t(u_n; b) = \left.\frac{\partial G}{\partial u}\right|_{(u_n;b)} \tag{21}$$

Note that the higher order stiffness matrix K_h is a function of u_n and Δu. In addition, the consistent tangent stiffness can be defined as

$$K_c(u_n, \Delta\tilde{u}; b) = \left.\frac{\partial \Delta G}{\partial \Delta u}\right|_{(u_n,\Delta\tilde{u};b)} = \left.\left(K_t + \frac{\partial (K_h \Delta u)}{\partial \Delta u}\right)\right|_{(u_n,\Delta\tilde{u};b)} \tag{22}$$

which will find application in the sensitivity calculation discussed later. It can also be shown that

$$K_c(u_n, \Delta\tilde{u}; b) = K_t(u_n, +\Delta\tilde{u}; b) \tag{23}$$

which is similar to Eq. (17). The incremental equilibrium equation, Eq. (20), needs to be solved together with a constraint equation associated with a chosen scheme. The system of equations including the equilibrium and constraint equations can be conveniently written in the $(N+1)$ dimensional space as[11]

$$\begin{bmatrix} K_t + K_h & -P_r \\ \langle C \rangle & \alpha \end{bmatrix} \begin{Bmatrix} \Delta u \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ \beta \end{Bmatrix} \tag{24}$$

where $\langle\cdot\rangle$ denotes a row vector, C refers to the constraint, and α and β are scalars. The corresponding $\langle C\rangle$, α, and β for each scheme are as follows:

NR:

$$\langle C \rangle = \langle 0 \rangle, \alpha = 1, \quad \beta = \text{prescribed } \Delta\lambda \tag{25a}$$

DC:

$$\langle C \rangle = \langle 0, \ldots, 0, 1, 0, \ldots 0 \rangle, \alpha = 0, \quad \beta = \text{prescribed } \Delta u_i \tag{25b}$$

ARC:

$$\langle C \rangle = \langle \Delta u \rangle, \alpha = \Delta\lambda, \beta = \Delta S^2; \Delta S = \text{ prescribed arc length} \tag{25c}$$

WC:

$$\langle C \rangle = \Delta\lambda \langle P_r \rangle, \alpha = 0,$$

$$\beta = \Delta W; \Delta W = \text{prescribed work increment} \tag{25d}$$

In Eq. (25b), the only nonzero element with value of 1 in $\langle C\rangle$ is the ith element, and Δu_i is the corresponding prescribed displacement increment. The procedure for solving Eq. (24) is standard and is not repeated here (e.g., Refs. 11 and 12). One should note that K_h is only used in the force recovery phase but not in the predictor phase. Also, the inclusion of K_h in geometric nonlinear analysis is important, as shown in Refs. 13 and 14.

The system of equations governing sensitivity can be obtained by differentiation of Eq. (24) with respect to b. This yields

$$\begin{bmatrix} K_c & -P_r \\ \langle C \rangle & \tilde{\alpha} \end{bmatrix} \begin{Bmatrix} \Delta \overset{*}{u} \\ \Delta\overset{*}{\lambda} \end{Bmatrix} + \begin{Bmatrix} P_s \\ 0 \end{Bmatrix} = 0 \tag{26}$$

where the definition of K_c in Eq. (22) has been used. In Eq. (26), the scalar $\tilde{\alpha}$ is the same as α defined in Eqs. 25a–25c. For the work control method, $\tilde{\alpha} = \langle P_r\rangle \Delta u$. The vector P_s is the so-called pseudoloading vector, which is defined as

$$P_s = \left[\left(\frac{\partial K_t}{\partial u_n} + \frac{\partial K_h}{\partial u_n}\right)\overset{*}{u}_n\right]\Delta u + \underline{\left(\frac{\partial K_t}{\partial b} + \frac{\partial K_h}{\partial b}\right)\Delta u} = P_{s1} + \underline{P_{s2}} \tag{27}$$

where the underlined parts of Eq. (27) are defined to be equal. Several points should be noted. First, the calculation of P_s is a postprocessing step because it is a function of only $\overset{*}{u}_n$, u_n, Δu, and b. After obtaining Δu by solving Eq. (24), P_s can be determined. Second, the second part of the pseudoloading vector P_{s2} is a function of only u_n, Δu, and b but not of $\overset{*}{u}_n$ or $\overset{*}{\lambda}_n$. Therefore, for any two schemes $M1$ and $M2$, the following identity holds

$$\begin{bmatrix} K_c & -P_r \\ \langle C \rangle & \tilde{\alpha} \end{bmatrix}_{M1} \begin{Bmatrix} \Delta \overset{*}{u} \\ \Delta\overset{*}{\lambda} \end{Bmatrix}_{M1} + \begin{Bmatrix} P_{s1} \\ 0 \end{Bmatrix}_{M1} = \begin{bmatrix} K_c & -P_r \\ \langle C \rangle & \tilde{\alpha} \end{bmatrix}_{M2} \begin{Bmatrix} \Delta \overset{*}{u} \\ \Delta\overset{*}{\lambda} \end{Bmatrix}_{M2} + \begin{Bmatrix} P_{s1} \\ 0 \end{Bmatrix}_{M2} \tag{28}$$

This equation can be used to relate sensitivities obtained from any two schemes. The procedure for calculating the equivalent $M1$ sensitivities from the $M2$ sensitivities is as follows.

1) Initialize the values of $\overset{*}{u}$ and $\overset{*}{\lambda}$ for $M2$ and the equivalent values of $\overset{*}{u}$ and $\overset{*}{\lambda}$ for $M1$; all of the values are zero at the beginning.
2) Solve Eq. (24) to obtain Δu and $\Delta\lambda$ for $M2$.
3) Calculate the system matrix of Eq. (26) for $M2$.
4) Evaluate P_{s1} and P_{s2} for $M2$ using Eq. (27).
5) Solve Eq. (26) to obtain $\Delta\overset{*}{u}$ and $\Delta\overset{*}{\lambda}$ for $M2$.
6) Evaluate the right-hand side of Eq. (28).
7) Calculate P_{s1} [Eq. (27)] for $M1$ by using the values of u_n, Δu of $M2$, and the equivalent $\overset{*}{u}_n$ of $M1$.
8) Calculate the system matrix of Eq. (28) for $M1$ (the forms of $\langle C\rangle$ and $\tilde{\alpha}$ should correspond to $M1$) by using the values of u_n, Δu, and $\Delta\lambda$ of $M2$.
9) Solve Eq. (28) to obtain the equivalent $M1$ values of $\Delta\overset{*}{u}$ and $\Delta\overset{*}{\lambda}$.
10) Update u_n, λ_n, $\overset{*}{u}_n$, and $\overset{*}{\lambda}_n$ for $M2$ and the equivalent $M1$ values of $\overset{*}{u}_n$ and $\overset{*}{\lambda}_n$.

Another way to relate sensitivities can be derived from differentiation of Eq. (20) with respect to b and noting the fact that $\partial G/\partial b$ are functions of only u and b. Therefore, for two schemes $M1$ and $M2$,

$$\left(\frac{\partial G}{\partial u}\overset{*}{u} - \overset{*}{\lambda} P_r\right)_{M1} = \left(\frac{\partial G}{\partial u}\overset{*}{u} - \overset{*}{\lambda} P_r\right)_{M2} \tag{29}$$

Indeed, this approach has been used before in the spring model [cf. Eq. (13)]. The main limitation of this approach is that the analytical form of G must be known. However, the analytical form of G very often cannot be obtained. Another limitation is that this approach cannot be used to relate sensitivities from two schemes if one of them is the arc length or work control method, as discussed in the spring model. To this end, the constraints need to be included, as in the derivation of Eq. (28).

Spring Model Revisited

The new approach of the previous section is applied to the spring model. In particular, the equivalent transformation between the sensitivities obtained from the arc length and work control methods is demonstrated explicitly. However, note that the approach is quite general in the sense that it can be used to relate sensitivities of any two schemes.

Using Eq. (28) for the arc length and work control methods, for example, one obtains

$$\begin{bmatrix} k_c & -1 \\ \Delta x & \Delta p \end{bmatrix}_{\text{ARC}} \begin{Bmatrix} \Delta\overset{*}{x} \\ \Delta\overset{*}{p} \end{Bmatrix}_{\text{ARC}} + \begin{Bmatrix} p_{s1} \\ 0 \end{Bmatrix}_{\text{ARC}} = \begin{bmatrix} k_c & -1 \\ \Delta p & \Delta x \end{bmatrix}_{\text{WC}} \begin{Bmatrix} \Delta\overset{*}{x} \\ \Delta\overset{*}{p} \end{Bmatrix}_{\text{WC}} + \begin{Bmatrix} p_{s1} \\ 0 \end{Bmatrix}_{\text{WC}} \tag{30}$$

Note that the reference load is taken as 1; therefore λ can be identified as p. Also, the first part of the pseudoloading, as defined in Eq. (27), can be obtained from Eq. (5) as

$$p_{s1} = 3k_1 \overset{*}{x}_n \Delta x (2x_n + \Delta x) \tag{31}$$

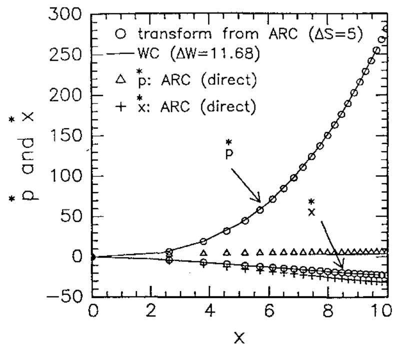

Fig. 4 Equivalent work control sensitivities from the arc length method compared with work control sensitivities for case 2; also the directly calculated sensitivities from the arc length method.

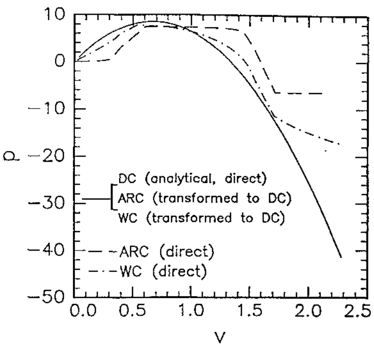

Fig. 5 Equivalent displacement control sensitivities for the symmetrical truss with height the design variable, also directly calculated sensitivities from the arc length and work control methods.

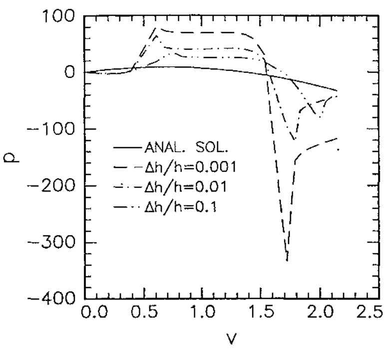

Fig. 6 Equivalent displacement control sensitivities obtained by using finite difference of two arc length solutions.

The numerical results obtained by using Eq. (30) and by following the procedure summarized in the last section, are depicted in Fig. 4 for case 2 ($b = k_1$). Figure 4 shows the equivalent work control sensitivities obtained from the sensitivities using the arc length method with $\Delta S = 5$ and those directly from the work control method with $\Delta W = 11.68$. These two results match perfectly; of course, they are equivalent to the analytical sensitivities as given in Eq. (9a) or (9b) if proper transformation is used. Note that the choice of $\Delta W = 11.68$ is calculated by using Eq. (4) and the first step solution of Δx and Δp from the arc length method. As discussed before, the sensitivities from the work control method depend on the choice of ΔW. Therefore, the use of an equivalent value of ΔW corresponding to ΔS is required. For comparison, the sensitivities directly calculated (untransformed) from the arc length method are also shown in Fig. 4.

Shallow Truss Example

More example to validate the proposed approach are provided for a shallow truss with two degrees of freedom, as shown in Fig. 1. This is a typical structure with snap-through (limit load) and snap-back responses. An updated Lagrangian finite element formulation is used to obtain the numerical solutions. The $\boldsymbol{K}_e$, $\boldsymbol{K}_g$, and $\boldsymbol{K}_h$ stiffness matrices for truss elements are readily available in Ref. 13 and are not repeated here. By using these matrices, the pseudoloading vector $\boldsymbol{P}_s$ in Eq. (27) is calculated by employing the analytical differentiation approach.

Here, for convenience, no physical units are used. Assume that both members have the same cross-sectional area $A = 1$, and also the same Young's modulus $E = 100{,}000$. Design variables include the height h, the area A, and the Young's modulus E. The initial sensitivities for θ_i^0 and $L_i (i = 1, 2$; cf. Fig. 1) can be obtained for each design variable as 1) if $b = A$ or E, then $\overset{*}{\theta}{}_i^0 = 0$ and $\overset{*}{L}_i = 0$; and 2) if $b = h$, then $\overset{*}{\theta}{}_i^0 = \cos\theta_i^0/L_i$ and $\overset{*}{L}_i = \sin\theta_i^0$. Condition 1 is obvious and Condition 2 can be obtained from the initial geometry.

The force equilibrium in the u and v directions, referred to Fig. 1, can be expressed as

$$G_1(u, v, p; b) = EA\left(\frac{l_1 - L_1}{L_1}\right)\frac{d_1 + u}{l_1} - EA\left(\frac{l_2 - L_2}{L_2}\right)\frac{d_2 - u}{l_2} = 0 \tag{32a}$$

$$G_2(u, v, p; b) = EA\left(\frac{l_1 - L_1}{L_1}\right)\frac{h - v}{l_1} + EA\left(\frac{l_2 - L_2}{L_2}\right)\frac{h - v}{l_2} = p \tag{32b}$$

where b stands for E, A, or h, and

$$l_1 = \sqrt{(d_1 + u)^2 + (h - v)^2}; \qquad l_2 = \sqrt{(d_2 - u)^2 + (h - v)^2}$$

are the current lengths of L_1 and L_2, respectively. Note that the analytical axial force for the updated Lagrangian formulation,[15] $F_i = EA(l_i - L_i)/L_i (i = 1, 2)$, has been used in Eqs. (32a) and (32b).

First, a symmetrical structure is considered with dimensions $d_1 = d_2 = 25$ and $h = 1$, subjected to downward loading. In this case, the analytical solution can be obtained by the displacement control method (as in the spring model). The (numerically calculated) equivalent displacement control design sensitivities $\overset{*}{p}$ with reference to h are shown in Fig. 5. It is clearly seen that all of the equivalent sensitivities are perfectly equivalent. The directly calculated (untransformed) sensitivities from the arc length and work control methods are also shown for comparison (here, only $\overset{*}{p}$ is shown, even though $\overset{*}{v}$ is not zero also). Interestingly, it is found that the finite difference method performs poorly in getting the sensitivities for the arc length and work control methods. For example, the equivalent displacement sensitivities obtained by using the sensitivities from the finite different method in conjunction with Eq. (29) [the corresponding component of $\boldsymbol{G}$ is given in Eq. (32b) with $u = 0$] are presented in Fig. 6. In this figure, two initial calculations are carried out using the arc length method. A wide range of perturbations Δh is tried here (in the figure, only three values are shown); however, the results are all very poor. It is also found that the errors (compared with the analytical differentiation approach using the FEM) of the finite difference sensitivities start increasing very quickly when the limit points are approached. Note that the limit points for this problem are located at $v = 0.423$ and 1.577. Similar phenomena occur for the work control method. The calculations demonstrate the inadequacy of using the difference of two neighboring solutions, at least for the arc length and work control methods, to calculate sensitivities for problems involving limit point response. However, for this example, the finite difference method generates very good sensitivity results corresponding to the displacement control method. These displacement control results, which are not shown here, are in good agreement with the analytical solutions shown in Fig. 5.

Further, an unsymmetrical structure with the dimensions of $d_1 = 20$, $d_2 = 30$, and $h = 1$ is used to test the approach developed in this study. Figure 7 shows the responses for the structural analysis problem. The results from all of the possible schemes (DC, ARC, and WC) are indistinguishable within plotting accuracy. It is useful

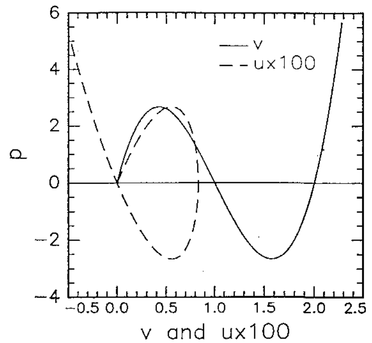

Fig. 7 Force-displacement responses for the unsymmetrical truss.

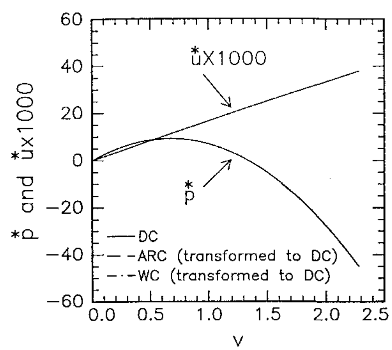

Fig. 8 Equivalent displacement control sensitivities for the unsymmetrical truss with height the design variable.

to note that when the original configuration (zero load state) in Fig. 7 reaches its mirror image, one has the exact solution $v = 2$, $u = 0$, and $p = 0$. Numerical results for this configuration, using the displacement control method, for example, are $v = 2, u = 6 \times 10^{-8}$, and $p = 9 \times 10^{-4}$. This serves as a check for the accuracy of the numerical results in Fig. 7. The equivalent displacement control sensitivities with respect to h are shown in Fig. 8. Again, all of the possible schemes (DC, ARC, and WC) give the same results within plotting accuracy. Note that these solutions are also checked against the results obtained by using Eq. (29), where the corresponding components of $\boldsymbol{G}$ are given in Eqs. (32a) and (32b). The final example is concerned with calculating the equivalent displacement control sensitivities (of the unsymmetrical truss) with respect to the cross-sectional area. It is found that the responses of all of the equivalent $\overset{*}{p}$ numerically obtained from all of the schemes are exactly the same as that of p in Fig. 7. This can also be concluded from Eqs. (32a) and (32b) by noting that here $A = 1$. Also, all of the equivalent $\overset{*}{u}$, which are not shown here, are zero for all of the schemes for this example. This can be confirmed by using Eq. (32a); suppose that (u, v) are the solutions for $b = A$, then obviously, they are also the solutions for $b = A + \Delta A$. The sensitivities with respect to the Young's modulus E are similar to the area sensitivities, as expected from Eqs. (32a) and (32b), and are not shown here.

Conclusions

Sensitivity results are scheme dependent due to different constraints associated with different schemes. However, based on physical grounds, these sensitivities must be equivalent. A new approach is developed to show this equivalence. Several examples are considered to test this new approach and excellent results are obtained in all cases.

It is also found that sensitivities corresponding to the arc length and work control methods cannot be obtained by using the finite difference method, in particular, for structures with snap-through and/or snap-back responses. However, the analytical differentiation approach, as employed in the present study, gives excellent results.

The results reported in this paper can be of great practical use. For example, one might desire sensitivities for one situation (say, load controlled) but can only calculate and/or measure these quantities for another situation (say, displacement controlled). Such issues have been discussed, in some detail, earlier in this paper.

Acknowledgments

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